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ON PSEUDO-AMENABILITY OF $C(X, A)$ FOR NORM IRREGULAR A.

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ABSTRACT. Let X be a compact Hausdorff space, we show that for a norm irregular Banach algebra A with a bounded approximate identity, if A has an aproximate diagonal which is bounded with respect to the multiplier norm on $A\hat{\otimes}A$, then $C(X, A)$ has an approximate diagonal.

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1. INTRODUCTION

The notion of amenability in Banach algebra was initiated by Johnson in [6]. Since then, amenability has become a major issue in Banach algebra theory and in harmonic analysis. For details on amenability in Banach algebras see [12] and for other notions of amenability in Banach algebras see [8, 10, 11, 12, 13, 14, 15, 16].

For a compact Hausdorff space X, Seinberg in [17] show that the Banach algebra $C(X)$ of all complex valued continuous functions on X with uniform norm is amenable. It is generally believed that the proof of Seinberg in [17] is too abstract in nature and so Abtahi and Zhang in [1] gave a new proof of the amenability of $C(X)$ using a characterization of amenability in terms of bounded approximate diagonal given by Johnson in [7] by constructing a bounded approximate diagonal for $C(X)$.

For a compact Hausdorff space X and a Banach algebra A, it is known that the algebraic properties of the Banach algebra $C(X, A)$ of all A-valued continuous functions on X are derived from those of A.

Ghamarshoushtari and Zhang in [5] showed that $C(X, A)$ being amenable is equivalent to the amenability of the Banach algebra A. It should be noted that the proof of Ghamarshoushtari and Zhang in [5] relied heavily on the boundedness condition of the approximate diagonal on A. A natural question of interest that naturally arises from the work of Gharmashoushtari and Zhang stated above is: Does the pseudo-amenability of $C(X, A)$ follow from that of the range algebra A? That is, can we remove the boundedness condition from the approximate diagonal on A?

In this work, we provide partial answer to the above question by studying the pseudo-amenability of $C(X, A)$ in terms of the pseudo-amenability of

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the Banach algebra A. In particular, we consider norm irregular Banach algebra A with a bounded approximate identity and assume that A has an aproximate diagonal which is bounded with respect to the multiplier norm on $A\hat{\otimes}A$ and show that $C(X, A)$ has an approximate diagonal, i.e. pseudo-amenable.

2. Preliminaries

First, we recall some standard notions, for further details, see [3] and [10].

Let A be a Banach algebra and E a Banach A-bimodule, a linear map $D : A \to E$ is called a derivation if

$$D(ab) = aD(b) + D(a)b \;\; (a, b \in A). \tag{1}$$

A derivation D is inner if there exists an $x \in E$ such that $D(a) = ax - xa$ for every $a \in A$.

Definition 2.1 A Banach algbebra A is said to be amenable if every continuous derivation $D : A \to E'$ is inner for every Banach A-bimodule E. An approximate diagonal for A is a net $(\alpha_\lambda) \subset A\hat{\otimes}A$ such that for any $a \in A$,

$$a.\alpha_\lambda - \alpha_\lambda.a \to 0 \text{ and } \pi(\alpha_\lambda)a \to a \text{ for all } \lambda,$$

where $\pi : A\hat{\otimes}A \to A$ is the diagonal operator determined by $\pi(a \otimes b) = ab \quad (a, b \in A)$. An approximate diagonal $(\alpha_\lambda) \subset A\hat{\otimes}A$ for A is said to be bounded if there exists a positive number K such that $\|\alpha_\lambda\|_p \leq K$ for all λ, where $\|.\|_p$ is the projective tensor norm on $A\hat{\otimes}A$.

Johnson in [7] showed that the amenability of the Banach algebra A is equivalent to A possessing a bounded approximate diagonal. Also, we recall from [4] that A is pseudo-amenable if it possess an approximate diagonal (not necessarily bounded). Thus, A being amenable implies it is pseudo-amenable.

For a compact Hausdorff space X, $C(X)$ denotes the algebra of all complex valued continuous functions on X. It is a commutative Banach algebra with pointwise product and uniform norm

$$\|f\|_\infty = \sup_{t\in X} |f(t)| \quad (f \in C(X)).$$

Also, for a Banach algebra A, $C(X, A)$ denotes the algebra of all A-valued continuous functions on X. It is a Banach algebra with pointwise product and uniform

$$\|f\|_\infty = \sup_{t\in X} \|f(t)\|_A \quad (f \in C(X, A)).$$

$C(X, A)$ is not commutative in general, it is commutative whenever A is commutative.

We recall from [2] that the multiplier semi norm $\|.\|_M$ on A is defined as:

$$\|a\|_M = \sup_{b\in A,\, \|b\|\leq 1} \{\|ab\|,\, \|ba\|\} \;\; (a \in A). \tag{2}$$

Clearly, $\max\{\|ab\|,\, \|ba\|\} \leq \|a\|_M\|b\|$ for all $a, b \in A$, so that $\|a\|_M \leq \|a\| \quad (a \in A)$. The annihilator ideal of A denoted by $ann(A)$ is defined as

$$ann(A) = \{a \in A : ab = ba = 0,\;\; b \in A\}. \tag{3}$$

If $ann(A) = \{0\}$, then $\|.\|_M$ is indeed an algebra norm on A called the multiplier norm.

Definition 2.2 A Banach algebra A is norm irregular if the multiplier norm on A is strictly weaker than the complete norm on A.
That is $\|a\|_M < \|a\| \quad (a \in A)$.
Since the projective tensor norm on $A\hat{\otimes}A$ depends on the norm on A, it follows that if $A\hat{\otimes}A$ is norm irregular, then A is also norm irregular.

3. Main Results

The following lemma is an important component of our main result.

Lemma 3.1. *Let A be a Banach algebra with a bounded approximate identity and let $(\|.\|_p)_M$ be the multiplier semi norm on $A\hat{\otimes}A$. Then there exists $K > 0$ such that*

$$\|a\beta\|_p \leq K(\|\beta\|_p)_M\|a\|,$$

for all $a \in \mathcal{A}$, $\beta \in \mathcal{A}\hat{\otimes}\mathcal{A}$.

Proof. Let (e_α) be a bounded approximate identity for A. Then there exists a $C > 0$ such that $\|e_\alpha\| \leq C$ for all α.
Case 1: Suppose $1 \leq \|e_\alpha\| \leq C$, for all α. Then for any $a \in A$ and $\beta \in A\hat{\otimes}A$,

$$\begin{aligned}\|a\beta\|_p &\leq \|(a \otimes e_\alpha)\beta\|_p \\ &\leq (\|\beta\|_p)_M\|a\|\|e_\alpha\| \\ &\leq K(\|\beta\|_p)_M\|a\|.\end{aligned}$$

Here, we chose $K = C$.
Case 2: Suppose $C < 1$, then $\|e_\alpha\| < 1$ for all α. It then follows that there exists $L > 1$ such that $L\|e_\alpha\| \geq 1$ for all α. For any $a \in A$ and $\beta \in A\hat{\otimes}A$,

$$\begin{aligned}\|a\beta\|_p &\leq L\|(a \otimes e_\alpha)\beta\|_p \\ &\leq L(\|\beta\|_p)_M\|a\|\|e_\alpha\| \\ &\leq LC(\|\beta\|_p)_M\|a\| \\ &= K(\|\beta\|_p)_M\|a\|,\end{aligned}$$

where $K = LC$. □

We now give our main result.

Theorem 3.2. *Let X be a compact Hausdorff space and $(A, \|.\|)$ be a Banach algebra with a bounded approximate identity such that $A\hat{\otimes}A$ is norm irregular. If A has an approximate diagonal which is bounded with respect to the multiplier norm on $A\hat{\otimes}A$, then $C(X, A)$ has an approximate diagonal and so it is pseudo-amenable.*

Proof. We define a linear map $T : (C(X)\hat{\otimes}C(X), A\hat{\otimes}A) \to C(X, A)\hat{\otimes}C(X, A)$ determined by $T(v, \beta) = \sum_{i,j} u_i\alpha_j \otimes v_i\beta_j$, where $v = \sum_i u_i \otimes v_i \in C(X) \otimes C(X)$ and $\beta = \sum_j \alpha_j \otimes \beta_j$. Notice that

$$\sum_{i,j} \|u_i\alpha_j\|\|v_j\beta_j\| \leq \sum_i \|u_i\|\|v_i\| \sum_j \|\alpha_j\|\|\beta_j\|.$$

It follows that $\|T(u,\beta)\|_p \leq \|u\|_p\|\beta\|_p$, where $\|.\|_p$ is the projective tensor norm. By the argument in [4, Proposition 2.1], we are to show that for every $\epsilon > 0$ and finite set $F \subset C(X,A)$ there exists a net $U = U_{(F,\epsilon)} \in C(X,A)\hat{\otimes}C(X,A)$ such that

(1) $\|g.U - U.g\|_p < \epsilon$, and
(2) $\|\pi(U)g - g\| < \epsilon$, for all $g \in F$.

Let (α_λ) be an approximate diagonal for A, bounded with respect to $(\|.\|_p)_M$. Then there exists $K_1 > 0$ such that $(\|\alpha_\lambda\|_p)_M \leq K_1$ for all λ. Let $\epsilon > 0$ and let $F \subset C(X,A)$ be a fixed finite subset. For fixed finite subsets F_C, F_A in $C(X)$ and A respectively, $\phi_j s \in F_C$ and $a_j s \in F_A$, we see that $\sum_j \phi_j a_j \in F$, where the sum is finite. Let L be a positive real number such that $\|a\| < L,\ a \in F_A$ and $\|\phi\| < L$ $(\phi \in F_C)$. Recall (α_λ) is an approximate diagonal for A. It then follows that for any $\alpha \in (\alpha_\lambda)$ and $b \in F_A$,

$$\|b.\alpha - \alpha.b\| < \frac{\epsilon}{8cNL},\ \ \|\pi(\alpha)b - b\| < \frac{\epsilon}{4cNL} \text{ for all } \alpha \in (\alpha_\lambda),$$

where N is a positive integer chosen to be no less than the number of terms in the finite sums $\sum_j \phi_j a_j$, and c is the constant in [5]. Since X is compact, there exist finite open sets $V_i \subset X,\ i = 1,2,..,n$ such that $X = \bigcup_{i=1}^{n} V_i$ and $|\phi(x) - \phi(y)| < \epsilon$ $(\phi \in F_C,\ x,y \in V_i)$. By applying partition of unity, we obtain continuous functions $h_i \in C(X)$, $i = 1,2,...,n$ such that $\text{supp}(h_i) \subset V_i$, $h_i \in [0,1]$ for all i and $\sum_i h_i = 1$ on X. Let $u_i = \sqrt{h_i}$ and $u = \sum_i^n u_i \otimes u_i$. Clearly, $\pi(u) = 1$ on X, $\|u\|_p \leq 2c$ and for every $\phi \in F_C$,

$$\|\phi.u - u.\phi\|_p < \frac{\epsilon}{4KK_1NL},$$

where K is the constant in Lemma 3.1. Since $lin\{\phi a : \phi \in C(X), a \in A\}$ is dense in $C(X,A)$, it then follows that for any $g \in F$,

$$\left\| g - \sum_j \phi_j a_j \right\|_\infty < \frac{\epsilon}{8\|\alpha\|_p c}.$$

Thus for any $\alpha \in (\alpha_\lambda)$,

$$\begin{aligned}
\left\| \sum_i \phi_j a_j T(u,\alpha) - T(u,\alpha)\sum_j \phi_j a_j \right\| &= \left\| \sum_j \phi_j a_j T(u,\alpha) - \sum_j T(u,\alpha)\phi_j a_j \right\| \\
&\leq \sum_j \|T(\phi_j u, a_j\alpha) - T(u\phi_j, \alpha a_j)\| \\
&\leq \sum_j \|T(\phi_j u, a_j\alpha) - T(u\phi_j, a_j\alpha)\| \\
&+ \sum_j \|T(u\phi_j, a_j\alpha) - T(u\phi_j, \alpha a_j)\| \\
&\leq \sum_j \|\phi_j u - u\phi_j\|_p \|a_j\alpha\|_p \\
&+ \sum_j \|u\phi_j\| \|a_j\alpha - \alpha a_j\|_p.
\end{aligned}$$

Since A has a bounded approximate identity, we apply Lemma 3.1 and obtain

$$\begin{aligned}
\sum_j \|\phi_j u - u\phi_j\|_p \|a_j\alpha\|_p + \sum_j \|u\phi_j\| \|a_j\alpha - \alpha a_j\|_p \\
&\le \sum_j \|\phi_j u - u\phi_j\|_p K\|a_j\|(\|\alpha\|_p)_M \\
&+ \sum_j \|u\|_p \|\phi_j\|_\infty \|a_j\alpha - \alpha a_j\|_p \\
&\le K(\|\alpha\|_p)_M \sum_j \|\phi_j u - u\phi_j\|_p \|a_j\| \\
&+ \|u\|_p \sum_j \|\phi_j\|_\infty \|a_j\alpha - \alpha a_j\|_p \\
&\le KK_1 L \frac{\epsilon}{4KK_1NL} N + 2cL\frac{\epsilon}{8cNL} N \\
&= \frac{\epsilon}{2}.
\end{aligned}$$

Set $f_N = \sum_j \phi_j a_j$, so that for any $g \in F$,

$$\begin{aligned}
\|gT(u,\alpha) - T(u,\alpha)g\| &= \|(g - f_N + f_N)T(u,\alpha) - T(u,\alpha)(g - f_N + f_N)\| \\
&\le \|(g - f_N)T(u,\alpha)\| + \|f_N T(u,\alpha) - T(u,\alpha)f_N\| \\
&+ \|T(u,\alpha)(g - f_N)\| \\
&\le 2\|T(u,\alpha)\| \|g - f_N\|_\infty + \|f_N T(u,\alpha) - T(u,\alpha)f_N\| \\
&\le 2\|u\|_p \|\alpha\|_p \|g - f_N\|_\infty + \|f_N T(u,\alpha) - T(u,\alpha)f_N\| \\
&\le 4c\|g - f_N\|_\infty \|\alpha\|_p + \|f_N T(u,\alpha) - T(u,\alpha)f_N\| \\
&< 4c\frac{\epsilon}{8\|\alpha\|_p c}\|\alpha\|_p + \frac{\epsilon}{2} = \epsilon.
\end{aligned}$$

Also,

$$\begin{aligned}
\left\|\pi(T(u,\alpha))\sum_j \phi_j a_j - \sum_j \phi_j a_j\right\| &= \left\|\pi(u)\pi(\alpha)\sum_j \phi_j a_j - \sum_j \phi_j a_j\right\| \\
&= \left\|\sum_j \phi_j(\pi(\alpha)a_j - a_j)\right\| \\
&\le \sum_j \|\phi_j\|_\infty \|\pi(\alpha)a_j - a_j\| \\
&< NL\frac{\epsilon}{4cNL} \le \frac{\epsilon}{2}.
\end{aligned}$$

Let f_N be as defined earlier. Without loss of generality we may suppose $\|\alpha\|_p \geq 1$. Then for any $g \in F$,

$$\begin{aligned}
\|\pi(T(u,\alpha))g - g\| &= \|\pi(T(u,\alpha))(g - f_N + f_N) - (g - f_N + f_N)\| \\
&= \|\pi(T(u,\alpha))(g - f_N) + \pi(T(u,\alpha))f_N - f_N - (g - f_N)\| \\
&\leq \|\pi(\alpha)(g - f_N)\| + \|\pi(T(u,\alpha))f_N - f_N\| + \|g - f_N\| \\
&\leq \|\pi(\alpha)\|_M \|g - f_N\|_\infty + \|\pi(T(u,\alpha))f_N - f_N\| + \|g - f_N\|_\infty \\
&\leq \|\alpha\|_p \|g - f_N\|_\infty + \|\pi(T(u,\alpha))f_N - f_N\| + \|g - f_N\|_\infty \\
&\leq 2\|\alpha\|_p \|g - f_N\|_\infty + \|\pi(T(u,\alpha))f_N - f_N\| \\
&< 2\|\alpha\|_p \frac{\epsilon}{8\|\alpha\|_p} + \frac{\epsilon}{2} < \epsilon.
\end{aligned}$$

We set $T(u,\alpha) = U$, the natural partial order $(F_1, \epsilon_1) \prec (F_2, \epsilon_2)$ if and only if $F_1 \subset F_2$, $\epsilon_1 \geq \epsilon_2$, ensures we obtain a net $(U_{(F,\epsilon)})$, which is the desired approximate diagonal for $C(X, A)$. □

Remark 3.3 Since A has a bounded approximate identity, we may assume that $(\pi(\alpha_\lambda))$ is a bounded approximate identity for A. This further implies that $(\pi(\alpha_\lambda))$ is a bounded approximate identity for $C(X, A)$, so that for any $g \in C(X, A)$, $\|\pi(T(u,\alpha)g - g\|_\infty = \|\pi(\alpha)g - g\|_\infty \to 0$, which serves as another proof for condition (2).

Remark 3.4 Clearly, the map $C(X, A) \to A$, $f \mapsto f(t_0)$, where t_0 is fixed in X is a continuous epimorphism. This shows that the converse of Theorem 3.2 clearly holds. That is, if $C(X, A)$ has an approximate diagonal, then A has an approximate diagonal.

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